Enough to Check Collatz Conjecture for $16k + 11$

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Abstract. We show that it is enough to check Collatz conjecture for the integers of the form $16k + 11$.

1 Introduction

Collatz conjecture, also known as the $3x + 1$ conjecture, is simply stated as follows. Let $f : \mathbb{N} \to \mathbb{N}$ be a function defined as

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even.} \\ 3n + 1 & \text{if } n \text{ is odd.} \end{cases}$$

Let $\mathcal{H}$ be the set of natural numbers such that repeatedly applying f to the numbers eventually leads to 1.

Definition 1. $\mathcal{H} = \{n \in \mathbb{N} \mid$ *there exists* m *such that* $f^m(n) = 1\}$.

Collatz conjecture asserts that the set $\mathcal{H}$ is equal to the set of all natural numbers.

Conjecture 1 (Collatz). $\mathcal{H} = \mathbb{N}$.

In this paper, we show that it is enough to check this conjecture only for numbers of the form $16k + 11$.

Theorem 1. *If* $\{16k + 11 \mid k \in \mathbb{N}\} \subset \mathcal{H}$*, then Collatz conjecture is true.*

To the best of our knowledge [1], our result has not been known before.

2 Proof

Theorem 1 follows from the following proposition:

Proposition 1. *Under the assumption of Theorem 1, if* $\{n \mid n < 2^m\} \subset \mathcal{H}$ *for some* m*, then* $\{n \mid n < 2^{m+1}\} \subset \mathcal{H}$.

To prove the proposition, we need the following lemmas.

Lemma 1. $n \in \mathcal{H} \iff f^m(n) \in \mathcal{H}$ *for some* $m \in \mathbb{N}$.

Proof. Follows from the definitions of f and $\mathcal{H}$.

Lemma 2.

1. $2n + 1 \in \mathcal{H} \implies 8n + 5 \in \mathcal{H}$.
2. $8n + 1 \in \mathcal{H} \implies 16n + 3 \in \mathcal{H}$.
3. $4n + 1 \in \mathcal{H} \implies 8n + 3 \in \mathcal{H}$.
4. $4n + 3 \in \mathcal{H} \implies 8n + 7 \in \mathcal{H}$.

Proof. We use Lemma 1 in the proof.

1. $f^2(2n + 1) = 3n + 2 = f^4(8n + 5)$.
2. $f^3(8n + 1) = 6n + 1$. On the other hand, $f^2(16n + 3) = 24n + 5$. It follows from Lemma 2.1.

3. We consider two cases when n is even and odd. For $n = 2p$, $4n+1 = 8p+1$. We use Lemma 2.2 to have $8n + 3 \in \mathcal{H}$. When $n = 2p + 1$, from the assumption $16p + 11 = 8n + 3 \in \mathcal{H}$.
4. Choose integer q and r so that
$$n + 1 = 2^r(2q + 1).$$
Then $4n + 3 = 2^{r+2}(2q + 1) - 1$. Now $f^2(4n + 3) = 2^{r+1} \cdot 3(2q + 1) - 1$ and we see that the exponent decreases by one and the remaning factor is still odd. Continue this process until the expononent becomes 1. By the assumption,
$$f^{2r+2}(4n + 3) = 2 \cdot 3^{r+1}(2q + 1) - 1 = 4 \cdot 3^{r+1}q + 2 \cdot 3^{r+1} - 1 \in \mathcal{H}.$$
Since this number is of the form $4n+1$, we use Lemma 2.3 to see that $4 \cdot 3^{r+1}(2q+1) - 1 \in \mathcal{H}$. On the other hand, $8n + 7 = 2^{r+3}(2q + 1) - 1$ and similarly we obtain $f^{2r+2}(8n + 7) = 4 \cdot 3^{r+1}(2q + 1) - 1$. By Lemma 1, it follows that $8n + 7 \in \mathcal{H}$.

Proof of Proposition 1. Let $A_m := \{n \mid n < 2^m\}$. Pick x from A_{m+1}. There are five cases.

1. $x = 2k$.
$f(x) = k \in A_m$.
2. $x = 8k + 1$.
Write k as $k = 2^s \cdot t$ with integer s and odd t. Then
$$8k + 1 = 2^{s+3}t + 1.$$
Now $f^3(8k + 1) = 2^{s+1} \cdot 3t + 1$. As in the proof of Lemma 2.4, every three step decreases the exponent by two and the remaining factor stays odd.
When s is even, we get $2 \cdot 3^{s/2+1}t + 1$ after $3 \cdot (s/2 + 1)$ steps. Since $3^{s/2+1}t \in A_m$ is odd, the proof follows from Lemma 2.3 and 2.4. When s is odd, we get $4 \cdot 3^{(s+1)/2}t + 1$ after $3(s + 1)/2$ steps. Since $3^{(s+1)/2}t \in A_m$ is odd, we use Lemma 2.1.
3. $x = 8k + 3$.
$4k + 1 \in A_m$ and use Lemma 2.3.
4. $x = 8k + 5$.
$2k + 1 \in A_m$ and use Lemma 2.1.
5. $x = 8k + 7$.
$4k + 3 \in A_m$ and use Lemma 2.4.

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References

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