Lecture 5 —
Design and Implementation of PLs

(1) Expressions

CSE307: Programming Languages

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Designing a Programming Language

We need to specify syntax and semantics of the language:

- Syntax: how to write programs
- Semantics: the meaning of the programs

Both are formally specified by inductive definitions.

Let: Our First Language

Syntax

$$
\begin{aligned}
P \;\to\;& E \\
E \;\to\;& n \\
\mid\;& x \\
\mid\;& E + E \\
\mid\;& E - E \\
\mid\;& \texttt{iszero } E \\
\mid\;& \texttt{if } E \texttt{ then } E \texttt{ else } E \\
\mid\;& \texttt{let } x = E \texttt{ in } E \\
\mid\;& \texttt{read}
\end{aligned}
$$

Examples

```
let x = 1 in x + 2

let x = 1
in let y = 2
   in x + y

let x = let y = 2
        in y + 1
in x + 3

let x = 1
in let y = 2
   in let x = 3
      in x + y
```

Examples

```
let x = 1
in let y = let x = 2
           in x + x
   in x + y

let x = 1
in let y = 2
   in if iszero (x - 1) then y - 1 else  y + 1

let x = 1
in let y = iszero x
   in x + y
```

Values and Environments

To define the semantics, we need to define **values** and **environments**.

- The set of values that the language manipulates:
  - `1+(2+3)`
  - `iszero 1, iszero (2-2)`
  - `if iszero 1 then 2 else 3`
- An environment is a variable-value mapping, which is needed to evaluate expressions with variables:
  - `x, y`
  - `x+1, x+(y-2)`
  - ```
    let x = read
    in let y = 2
       in if zero x then y else x
    ```

Values and Environments

In Let, the set of values includes integers and booleans:

$$v \in Val = \mathbb{Z} + Bool$$

and an environment is a function from variables to values:

$$\rho \in Env = Var \rightarrow Val$$

Notations:

- []: the empty environment.
- $[x \mapsto v]\rho$ (or $\rho[x \mapsto v]$): the extension of ρ where x is bound to v:

$$([x \mapsto v]\rho)(y) = \begin{cases} v & \text{if } x = y \\ \rho(y) & \text{otherwise} \end{cases}$$

For simplicity, we write $[x_1 \mapsto v_1, x_2 \mapsto v_2]\rho$ for the extension of ρ where x_1 is bound to v_1, x_2 to v_2:

$$[x_1 \mapsto v_1, x_2 \mapsto v_2]\rho = [x_1 \mapsto v_1]([x_2 \mapsto v_2]\rho)$$

Evaluation of Expressions

Given an environment ρ, an expression e evaluates to a value v:

$$\rho \vdash e \Rightarrow v$$

or does not evaluate to any value (i.e. e does not have semantics w.r.t ρ).

- $[] \vdash 1 \Rightarrow 1$
- $[x \mapsto 1] \vdash \text{x+1} \Rightarrow 2$
- $[] \vdash \text{read} \Rightarrow 3$, $[x \mapsto 1] \vdash \text{read} \Rightarrow 5$
- $[x \mapsto 0] \vdash$ let y = 2 in if iszero x then y else x $\Rightarrow 2$

- iszero (iszero 3)
- if 1 then 2 else 3

Evaluation Rules

$$\boxed{\rho \vdash e \Rightarrow v}$$

$$\overline{\rho \vdash n \Rightarrow n} \qquad \overline{\rho \vdash x \Rightarrow \rho(x)}$$

$$\frac{\rho \vdash E_1 \Rightarrow n_1 \quad \rho \vdash E_2 \Rightarrow n_2}{\rho \vdash E_1 + E_2 \Rightarrow n_1 + n_2} \qquad \frac{\rho \vdash E_1 \Rightarrow n_1 \quad \rho \vdash E_2 \Rightarrow n_2}{\rho \vdash E_1 - E_2 \Rightarrow n_1 - n_2}$$

$$\overline{\rho \vdash \texttt{read} \Rightarrow n} \qquad \frac{\rho \vdash E \Rightarrow 0}{\rho \vdash \texttt{iszero } E \Rightarrow \textit{true}} \qquad \frac{\rho \vdash E \Rightarrow n}{\rho \vdash \texttt{iszero } E \Rightarrow \textit{false}} \ n \neq 0$$

$$\frac{\rho \vdash E_1 \Rightarrow \textit{true} \quad \rho \vdash E_2 \Rightarrow v}{\rho \vdash \texttt{if } E_1 \texttt{ then } E_2 \texttt{ else } E_3 \Rightarrow v} \qquad \frac{\rho \vdash E_1 \Rightarrow \textit{false} \quad \rho \vdash E_3 \Rightarrow v}{\rho \vdash \texttt{if } E_1 \texttt{ then } E_2 \texttt{ else } E_3 \Rightarrow v}$$

$$\frac{\rho \vdash E_1 \Rightarrow v_1 \quad [x \mapsto v_1]\rho \vdash E_2 \Rightarrow v}{\rho \vdash \texttt{let } x = E_1 \texttt{ in } E_2 \Rightarrow v}$$

Evaluation Rules

More precise interpretation of the evaluation rules:

- The inference rules define a set S of triples (ρ, e, v). For readability, the triple was written by $\rho \vdash e \Rightarrow v$ in the rules.

- We say an expression e has semantics w.r.t. ρ iff there is a triple $(\rho, e, v) \in S$ for some value v.

- That is, we say an expression e has semantics w.r.t. ρ iff we can derive $\rho \vdash e \Rightarrow v$ for some value v by applying the inference rules.

- We say an initial program e has semantics if $[] \vdash e \Rightarrow v$ for some v.

$$\frac{\dfrac{}{[] \vdash 1 \Rightarrow 1} \quad \dfrac{\dfrac{}{[x \mapsto 1] \vdash x \Rightarrow 1} \quad \dfrac{}{[x \mapsto 1] \vdash 2 \Rightarrow 2}}{[x \mapsto 1] \vdash x + 2 \Rightarrow 3}}{[] \vdash \texttt{let x = 1 in x + 2} \Rightarrow 3}$$

Examples

$$[y \mapsto 2, x \mapsto 1] \vdash x \Rightarrow 1$$
$$[y \mapsto 2, x \mapsto 1] \vdash y \Rightarrow 2$$

$$\frac{[x \mapsto 1] \vdash 2 \Rightarrow 2 \quad \overline{[y \mapsto 2, x \mapsto 1] \vdash x + y \Rightarrow 3}}{}$$

$$\frac{[] \vdash 1 \Rightarrow 1 \quad \overline{[x \mapsto 1] \vdash \texttt{let y = 2 in x + y} \Rightarrow 3}}{[] \vdash \texttt{let x = 1 in let y = 2 in x + y} \Rightarrow 3}$$

Examples

$$\frac{[] \vdash 2 \Rightarrow 2 \quad [y \mapsto 2] \vdash \text{y + 1} \Rightarrow 3 \quad [x \mapsto 3] \vdash \text{x} \Rightarrow 3 \quad [x \mapsto 3] \vdash 3 \Rightarrow 3}{\frac{[] \vdash \text{let y = 2 in y + 1} \Rightarrow 3 \qquad [x \mapsto 3] \vdash \text{x + 3} \Rightarrow 6}{[] \vdash \text{let x = (let y = 2 in y + 1) in x + 3} \Rightarrow 6}}$$

Examples

$$
\cfrac{
  [] \vdash 1 \Rightarrow 1
  \qquad
  \cfrac{
    [x \mapsto 1] \vdash 2 \Rightarrow 2
    \qquad
    \cfrac{
      [y \mapsto 2, x \mapsto 1] \vdash 3 \Rightarrow 3
      \qquad
      \cfrac{
        [y \mapsto 2, x \mapsto 3] \vdash x \Rightarrow 3
        \quad
        [y \mapsto 2, x \mapsto 3] \vdash y \Rightarrow 2
      }{
        [y \mapsto 2, x \mapsto 3] \vdash x + y \Rightarrow 5
      }
    }{
      [y \mapsto 2, x \mapsto 1] \vdash \texttt{let x = 3 in x + y} \Rightarrow 5
    }
  }{
    [x \mapsto 1] \vdash \texttt{let y = 2 in let x = 3 in x + y} \Rightarrow 5
  }
}{
  [] \vdash \texttt{let x = 1 in let y = 2 in let x = 3 in x + y} \Rightarrow 5
}
$$

Examples

$$
\dfrac{
  \dfrac{
    \dfrac{[x \mapsto 2] \vdash x \Rightarrow 2 \quad [x \mapsto 2] \vdash x \Rightarrow 2}{[x \mapsto 2] \vdash x \texttt{ + } x \Rightarrow 4}
  }{[x \mapsto 1] \vdash \texttt{let x = 2 in x + x} \Rightarrow 4}
  \quad
  \dfrac{
    \begin{array}{c} [y \mapsto 4, x \mapsto 1] \vdash x \Rightarrow 1 \\ [y \mapsto 4, x \mapsto 1] \vdash y \Rightarrow 4 \end{array}
  }{[y \mapsto 4, x \mapsto 1] \vdash x \texttt{ + } y \Rightarrow 5}
}{
  \dfrac{[x \mapsto 1] \vdash \texttt{let y = (let x = 2 in x + x) in x + y} \Rightarrow 5}{[] \vdash \texttt{let x = 1 in let y = (let x = 2 in x + x) in x + y } \Rightarrow 5}
}
$$

Examples

When ρ is $[x \mapsto 1, y \mapsto 2]$:

$$\frac{\dfrac{\rho \vdash x \Rightarrow 1 \quad \rho \vdash 1 \Rightarrow 1}{\rho \vdash x\ \text{-}\ 1 \Rightarrow 0}}{\rho \vdash \text{iszero}\ (x\ \text{-}\ 1) \Rightarrow \textit{true}} \quad \frac{\rho \vdash y \Rightarrow 2 \quad \rho \vdash 1 \Rightarrow 1}{\rho \vdash y\ \text{-}\ 1 \Rightarrow 1}$$
$$\overline{\rho \vdash \text{if iszero}\ (x\ \text{-}\ 1)\ \text{then}\ y\ \text{-}\ 1\ \text{else}\ y\ \text{+}\ 1 \Rightarrow 1}$$

Implementation of the Language

Syntax definition in OCaml:

```
type program = exp
and exp =
  | CONST of int
  | VAR of var
  | ADD of exp * exp
  | SUB of exp * exp
  | READ
  | ISZERO of exp
  | IF of exp * exp * exp
  | LET of var * exp * exp
and var = string
```

Example

```
let x = 7
in let y = 2
   in let  y = let x = x - 1
                in x - y
      in (x-8)-y

LET ("x", CONST 7,
   LET ("y", CONST 2,
      LET ("y", LET ("x", SUB(VAR "x", CONST 1),
                   SUB (VAR "x", VAR "y")),
        SUB (SUB (VAR "x", CONST 8), VAR "y"))))
```

Values and Environments

Values:

```
type value = Int of int | Bool of bool
```

Environments:

```
type env = (var * value) list
let empty_env = []
let extend_env (x,v) e = (x,v)::e
let rec apply_env x e =
  match e with
  | [] -> raise (Failure ("variable " ^ x ^ " not found"))
  | (y,v)::tl -> if x = y then v else apply_env x tl
```

Evaluation Rules

```
let rec eval : exp -> env -> value
=fun exp env ->
  match exp with
  | CONST n -> Int n
  | VAR x -> apply_env env x
  | ADD (e1,e2) ->
    let v1 = eval e1 env in
    let v2 = eval e2 env in
      (match v1,v2 with
      | Int n1, Int n2 -> Int (n1 + n2)
      | _ -> raise (Failure "Type Error: non-numeric values"))
  | SUB (e1,e2) ->
    let v1 = eval e1 env in
    let v2 = eval e2 env in
      (match v1,v2 with
      | Int n1, Int n2 -> Int (n1 - n2)
      | _ -> raise (Failure "Type Error: non-numeric values"))
  ...
```

Implementation: Semantics

```
let rec eval : exp -> env -> value
=fun exp env ->
  ...
  | READ -> Int (read_int())
  | ISZERO e ->
    (match eval e env with
    | Int n when n = 0 -> Bool true
    | _ -> Bool false)
  | IF (e1,e2,e3) ->
    (match eval e1 env with
    | Bool true -> eval e2 env
    | Bool false -> eval e3 env
    | _ -> raise (Failure "Type Error: condition must be Bool type"))
  | LET (x,e1,e2) ->
    let v1 = eval e1 env in
      eval e2 (extend_env (x,v1) env)
```

Interpreter

```
let run : program -> value
=fun pgm -> eval pgm empty_env
```

Examples:

```
# let e1 = LET ("x", CONST 1, ADD (VAR "x", CONST 2));;
val e1 : exp = LET ("x", CONST 1, ADD (VAR "x", CONST 2))
# run e1;;
- : value = Int 3
```

Summary

We have designed and implemented our first programming language:

$$
\begin{aligned}
P \;\rightarrow\; & E \\
E \;\rightarrow\; & n \\
\mid\; & x \\
\mid\; & E + E \\
\mid\; & E - E \\
\mid\; & \texttt{iszero } E \\
\mid\; & \texttt{if } E \texttt{ then } E \texttt{ else } E \\
\mid\; & \texttt{let } x = E \texttt{ in } E
\end{aligned}
$$

- key concepts: syntax, semantics, interpreter