Lecture 1 — Inductive Definitions (1)

CSE307: Programming Languages

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Inductive Definitions

Inductive definition (induction) is widely used in the study of programming languages and computer science in general: e.g.,

- The syntax and semantics of programming languages
- Data structures (e.g., lists, trees, graphs)

Induction is a technique for formally defining a set:

- The set is defined in terms of itself.
- The only way of defining an infinite set by a finite means.

Three styles to inductive definition:

- Top-down
- Bottom-up
- Rules of inference

Example (Top-Down)

Let us define a certain subset $\boldsymbol{S}$ of natural numbers ($\mathbb{N}$) as follows:

Definition ($\boldsymbol{S}$)

A natural number $\boldsymbol{n}$ is in $\boldsymbol{S}$ if and only if

1. $\boldsymbol{n} = 0$, or
2. $\boldsymbol{n} - 3 \in \boldsymbol{S}$.

The definition is *inductive*, because the set is defined in terms of itself. What is the set $\boldsymbol{S}$?

Example (Continued)

Let us see what natural numbers are in S.

- 0 is in S because of the first condition of the definition.
- 3 is in S because $3 - 3 = 0$ and 0 is in S.
- 6 is in S because $6 - 3 = 3$ and 3 is in S.
- $\ldots$

We can conjecture that $\{0, 3, 6, 9, \ldots\} \subseteq S$.

Proof by mathematical induction .

We show that $3k \in S$ for all $k \in \mathbb{N}$.

1. Base case: $3k \in S$ when $k = 0$.
2. Inductive case: Assume $3k \in S$ (Induction Hypothesis, I.H.).
   Then show $3 \cdot (k + 1) \in S$, which holds because $3 \cdot (k + 1) - 3 = 3k \in S$ by the induction hypothesis.

$\square$

Example (Continued)

What about other numbers? Does $\boldsymbol{S}$ contain only the multiples of 3?

- For instance, $1 \in \boldsymbol{S}$? No. Because the first condition is not true, the second condition must be true for 1 to be in $\boldsymbol{S}$. However, it is not true because $1 - 3 = -2$ is not a natural number. Similarly, we can show that $2 \notin \boldsymbol{S}$.
- What about 4? Because $4 - 3 = 1 \notin \boldsymbol{S}$, $4 \notin \boldsymbol{S}$.

By similar reasoning, we can conjecture that if $\boldsymbol{n}$ is not a multiple of 3 then $\boldsymbol{n}$ is not in $\boldsymbol{S}$. In other words, $\boldsymbol{S}$ contains multiples of 3 only: i.e.,

$$\{0, 3, 6, 9, \dots\} \supseteq \boldsymbol{S}.$$

Proof by contradiction.

Let $\boldsymbol{n} = 3\boldsymbol{k} + \boldsymbol{q}$ ($\boldsymbol{q} = 1$ or 2) and assume $\boldsymbol{n} \in \boldsymbol{S}$. By the definition of $\boldsymbol{S}$, $\boldsymbol{n} - 3$, $\boldsymbol{n} - 6$, $\dots$, $\boldsymbol{n} - 3\boldsymbol{k} \in \boldsymbol{S}$. Thus, $\boldsymbol{S}$ must include 1 or 2, a contradiction. □

A Bottom-up Definition

An alternative inductive definition of S:

Definition (S)

S is the *smallest* set such that $S \subseteq \mathbb{N}$ and S satisfies the following two conditions:

1. $0 \in S$, and
2. if $n \in S$, then $n + 3 \in S$.

- The two conditions imply $\{0, 3, 6, 9, \ldots\} \subseteq S$.
- The two conditions do not imply $\{0, 3, 6, 9, \ldots\} \supseteq S$. E.g.,
  - $\mathbb{N}$ satisfies the conditions: $0 \in \mathbb{N}$ and if $n \in \mathbb{N}$ then $n + 3 \in \mathbb{N}$.
  - $\{0, 3, 6, 9, \ldots\} \cup \{1, 4, 7, 10, \ldots\}$ satisfies the conditions.
- This is why the definition requires S to be the **smallest** such a set.
- The smallest set that satisfies the two conditions is unique:

$$S = \{0, 3, 6, 9, \ldots\}.$$

Rules of Inference

The third way is to define the set with inference rules. An inference rule is of the form:

$$\frac{A}{B}$$

- A: hypothesis (antecedent)
- B: conclusion (consequent)
- "if A is true then B is also true".
- $\overline{B}$: axiom (inference rule without hypothesis)

The hypothesis may contain multiple statements:

$$\frac{A \quad B}{C}$$

"If both A and B are true then so is C".

Rules of Inferences

The set $\boldsymbol{S}$ is defined as inference rules as follows:

Definition ($\boldsymbol{S}$)

$$\frac{}{0 \in \boldsymbol{S}} \qquad \frac{\boldsymbol{n} \in \boldsymbol{S}}{(\boldsymbol{n}+3) \in \boldsymbol{S}}$$

Interpret the rules as follows:

"A natural number $\boldsymbol{n}$ is in $\boldsymbol{S}$ iff $\boldsymbol{n} \in \boldsymbol{S}$ can be derived from the axiom by applying the inference rules finitely many times"

For example, $3 \in \boldsymbol{S}$ because we can find a "proof/derivation tree":

$$\cfrac{\cfrac{}{0 \in \boldsymbol{S}}\ \text{the axiom}}{3 \in \boldsymbol{S}}\ \text{the second rule}$$

but $1, 2, 4, \cdots \notin \boldsymbol{S}$ because we cannot find proofs. Note that this interpretation enforces that $\boldsymbol{S}$ is the smallest set closed under the inference rules.

Exercises

1. What set is defined by the following inductive rules?

$$\frac{}{3} \qquad \frac{x \quad y}{x+y}$$

2. What set is defined by the following inductive rules?

$$\frac{}{()} \qquad \frac{x}{(x)} \qquad \frac{x \quad y}{xy}$$

3. Define the following set as rules of inference:

$$S = \{a, b, aa, ab, ba, bb, aaa, aab, aba, abb, baa, bab, bba, bbb, \ldots\}$$

4. Define the following set as rules of inference:

$$S = \{x^n y^{n+1} \mid n \in \mathbb{N}\}$$

Summary

In inductive definitions, a set is defined in terms of itself. Three styles:

- Top-down
- Bottom-up
- Rules of inference

In PL, we mainly use the rules-of-inference method.