Lecture 4 — Nondeterministic Finite Automata

CSE205: Theory of Computation

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Definition

Definition (NFA)

A *nondeterministic finite automaton* (NFA) is a 5-tuple

$$\boldsymbol{M} = (\boldsymbol{Q}, \Sigma, \boldsymbol{\delta}, \boldsymbol{q}_0, \boldsymbol{F})$$

where

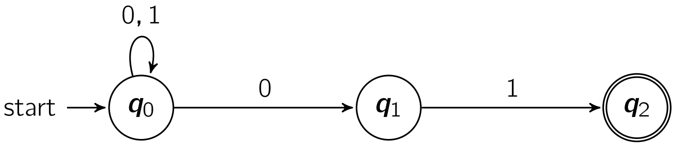
- $\boldsymbol{Q}$: a finite set of states
- Σ: a finite set of input symbols (the input alphabet)
- $\boldsymbol{q}_0 \in \boldsymbol{Q}$: the initial state
- $\boldsymbol{F} \subseteq \boldsymbol{Q}$: a set of final states
- $\boldsymbol{\delta} : \boldsymbol{Q} \times \Sigma \to 2^{\boldsymbol{Q}}$: transition function

The key difference from a DFA: $\boldsymbol{\delta}$ maps to a *set* of states, so a state may have zero, one, or many transitions on a symbol.

Example

$$(\{q_0, q_1, q_2\},\ \{0,1\},\ \delta,\ q_0,\ \{q_2\})$$

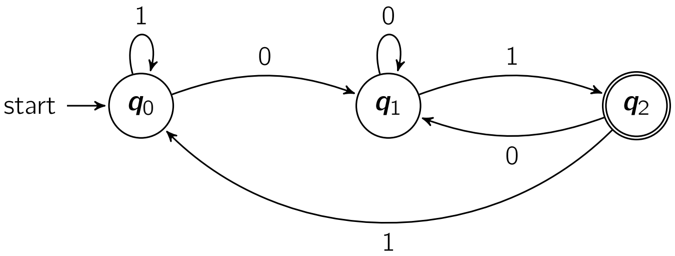
$$\begin{array}{ll}
\delta(q_0, 0) = \{q_0, q_1\} & \delta(q_0, 1) = \{q_0\} \\
\delta(q_1, 0) = \emptyset & \delta(q_1, 1) = \{q_2\} \\
\delta(q_2, 0) = \emptyset & \delta(q_2, 1) = \emptyset
\end{array}$$



This NFA accepts $\{w \in \{0,1\}^* \mid w \text{ ends in } 01\}$.

Example: Equivalent DFA

For comparison, here is an equivalent DFA for “ends in 01”.



The NFA is smaller and easier to design; the DFA must specify exactly one next state for every symbol.

Extended Transition Function

$$\hat{\delta} : Q \times \Sigma^* \to 2^Q$$

- **(Basis)** $s = \varepsilon$:

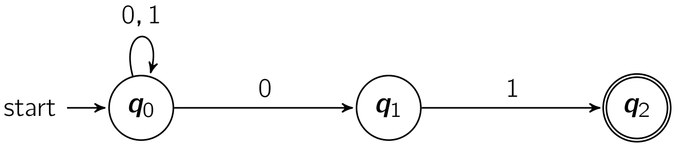
$$\hat{\delta}(q, \varepsilon) = \{q\}$$

- **(Induction)** $s = wa$:

$$\hat{\delta}(q, wa) = \bigcup_{s_i \in \hat{\delta}(q,w)} \delta(s_i, a)$$

We follow *all* possible paths and collect every state reachable on the input w.

Example



Computing $\hat{\delta}(q_0, 00101)$ step by step:

$$
\begin{aligned}
\hat{\delta}(q_0, \varepsilon) &= \{q_0\} \\
\hat{\delta}(q_0, 0) &= \delta(q_0, 0) = \{q_0, q_1\} \\
\hat{\delta}(q_0, 00) &= \delta(q_0, 0) \cup \delta(q_1, 0) = \{q_0, q_1\} \cup \emptyset = \{q_0, q_1\} \\
\hat{\delta}(q_0, 001) &= \delta(q_0, 1) \cup \delta(q_1, 1) = \{q_0\} \cup \{q_2\} = \{q_0, q_2\} \\
\hat{\delta}(q_0, 0010) &= \delta(q_0, 0) \cup \delta(q_2, 0) = \{q_0, q_1\} \cup \emptyset = \{q_0, q_1\} \\
\hat{\delta}(q_0, 00101) &= \delta(q_0, 1) \cup \delta(q_1, 1) = \{q_0\} \cup \{q_2\} = \{q_0, q_2\}
\end{aligned}
$$

Since $\{q_0, q_2\} \cap \{q_2\} \neq \emptyset$, the string 00101 is accepted.

Language of an NFA

Definition

The language of an NFA $M = (Q, \Sigma, \delta, q_0, F)$ is defined as

$$L(M) = \{w \in \Sigma^* \mid \hat{\delta}(q_0, w) \cap F \neq \emptyset\}.$$

A string is accepted if *at least one* of the reachable states after reading w is a final state.

Exercises

Design NFAs for the following languages:

1. $L = \{a^n b \mid n \geq 0\}$
2. $L = \{x01y \mid x, y \in \{0, 1\}^*\}$
3. $L = \{01w \mid w \in \{0, 1\}^*\}$
4. $L = \{w \in \{0, 1\}^* \mid w \text{ contains at least two 0's}\}$
5. $L = \{w \in \{0, 1\}^* \mid w \text{ contains exactly two 0's}\}$
6. $L = \{w \in \{0, 1\}^* \mid w \text{ has three consecutive 0's}\}$

Equivalence of DFA and NFA

Theorem (Equivalence)

A language $\boldsymbol{L}$ is accepted by some NFA if and only if $\boldsymbol{L}$ is accepted by some DFA.

Proof.

By the two lemmas below. $\square$

Lemma (DFA to NFA)

Given a DFA $\boldsymbol{D}$, there always exists an NFA $\boldsymbol{N}$ such that $\boldsymbol{L}(\boldsymbol{D}) = \boldsymbol{L}(\boldsymbol{N})$.

Lemma (NFA to DFA)

Given an NFA $\boldsymbol{N}$, there always exists a DFA $\boldsymbol{D}$ such that $\boldsymbol{L}(\boldsymbol{N}) = \boldsymbol{L}(\boldsymbol{D})$.

DFA to NFA

Lemma

Given a DFA D, there exists an NFA N such that $L(D) = L(N)$.

Proof. Assume a DFA $D = (Q, \Sigma, \delta_D, q_0, F)$ is given. Define an NFA

$$N = (Q, \Sigma, \delta_N, q_0, F), \qquad \delta_N(q, a) = \{\delta_D(q, a)\}.$$

It is enough to show $\hat{\delta}_N(q_0, w) = \{\hat{\delta}_D(q_0, w)\}$, by induction on $|w|$.

- $w = \varepsilon$: $\hat{\delta}_D(q_0, \varepsilon) = q_0$ and $\hat{\delta}_N(q_0, \varepsilon) = \{q_0\}$.
- $w = sa$:

$$\begin{aligned}\hat{\delta}_N(q_0, sa) &= \bigcup_{s_i \in \hat{\delta}_N(q_0, s)} \delta_N(s_i, a) = \delta_N(\hat{\delta}_D(q_0, s), a) \\ &= \{\delta_D(\hat{\delta}_D(q_0, s), a)\} = \{\hat{\delta}_D(q_0, sa)\}.\end{aligned}$$

NFA to DFA (Subset Construction)

Lemma

Given an NFA $\boldsymbol{N}$, there exists a DFA $\boldsymbol{D}$ such that $\boldsymbol{L}(\boldsymbol{N}) = \boldsymbol{L}(\boldsymbol{D})$.

Proof. Assume an NFA $\boldsymbol{N} = (\boldsymbol{Q_N}, \Sigma, \boldsymbol{\delta_N}, \boldsymbol{q}_0, \boldsymbol{F_N})$. Define a DFA

$$\boldsymbol{D} = (\boldsymbol{Q_D}, \Sigma, \boldsymbol{\delta_D}, \{\boldsymbol{q}_0\}, \boldsymbol{F_D})$$

where

- $\boldsymbol{Q_D} = 2^{\boldsymbol{Q_N}}$
- $\boldsymbol{F_D} = \{\boldsymbol{S} \in \boldsymbol{Q_D} \mid \boldsymbol{S} \cap \boldsymbol{F_N} \neq \emptyset\}$
- for each $\boldsymbol{S} \in \boldsymbol{Q_D}$ and $\boldsymbol{a} \in \Sigma$:

$$\boldsymbol{\delta_D}(\boldsymbol{S}, \boldsymbol{a}) = \bigcup_{\boldsymbol{p} \in \boldsymbol{S}} \boldsymbol{\delta_N}(\boldsymbol{p}, \boldsymbol{a})$$

NFA to DFA: Correctness

We prove $L(N) = L(D)$ by showing

$$\hat{\delta}_D(\{q_0\}, w) = \hat{\delta}_N(q_0, w),$$

by induction on $|w|$.

- $w = \varepsilon$: $\hat{\delta}_D(\{q_0\}, \varepsilon) = \{q_0\} = \hat{\delta}_N(q_0, \varepsilon)$.
- $w = sa$, with I.H. $\hat{\delta}_D(\{q_0\}, s) = \hat{\delta}_N(q_0, s)$:

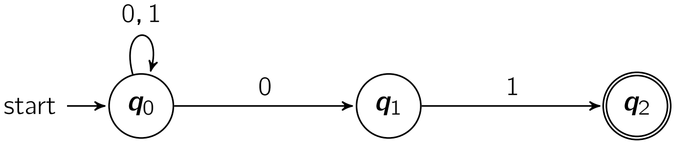
$$\begin{aligned}
\hat{\delta}_D(\{q_0\}, sa) &= \delta_D(\hat{\delta}_D(\{q_0\}, s), a) \\
&= \delta_D(\hat{\delta}_N(q_0, s), a) \\
&= \bigcup_{p \in \hat{\delta}_N(q_0, s)} \delta_N(p, a) = \hat{\delta}_N(q_0, sa).
\end{aligned}$$

Example: Subset Construction

Find a DFA equivalent to

$$N = (\{q_0, q_1, q_2\},\ \{0, 1\},\ \delta,\ q_0,\ \{q_2\})$$

$$\begin{array}{ll} \delta(q_0, 0) = \{q_0, q_1\} & \delta(q_0, 1) = \{q_0\} \\ \delta(q_1, 0) = \emptyset & \delta(q_1, 1) = \{q_2\} \\ \delta(q_2, 0) = \emptyset & \delta(q_2, 1) = \emptyset \end{array}$$



Example: Subset Construction

$$D = (Q_D,\ \{0,1\},\ \delta_D,\ \{q_0\},\ F_D)$$

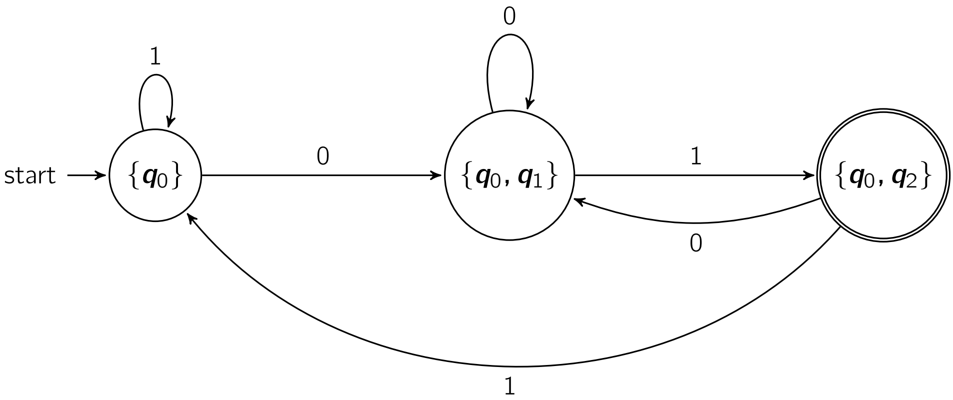
- $Q_D = 2^{\{q_0,q_1,q_2\}} = \{\emptyset, \{q_0\}, \{q_1\}, \ldots, \{q_0, q_1, q_2\}\}$
- $F_D = \{\{q_2\}, \{q_0, q_2\}, \{q_1, q_2\}, \{q_0, q_1, q_2\}\}$

Only the states reachable from $\{q_0\}$ matter:

δ_D	0	1
$\rightarrow \{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$
$\{q_0, q_1\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$
$*\{q_0, q_2\}$	$\{q_0, q_1\}$	$\{q_0\}$

Example: Subset Construction

The resulting DFA (only reachable states shown):



This DFA accepts exactly $\{w \mid w \text{ ends in } 01\}$, the same language as N.