Lecture 1 — Mathematical Preliminaries

CSE205: Theory of Computation

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2026 Fall

Today

- Mathematical backgrounds and notation
  - Sets
  - Inductive definitions and inductive proofs

Sets

A *set* is a collection of elements, e.g.,

$$\boldsymbol{S} = \{0, 1, 2\} = \{\boldsymbol{x} \in \mathbb{N} \mid 0 \leq \boldsymbol{x} \leq 2\}$$

$$\boldsymbol{S} = \{0, 2, 4, 6, \ldots\} = \{\boldsymbol{x} \in \mathbb{N} \mid \boldsymbol{x} \text{ is even}\}$$

Notations:

- $\emptyset$: the empty set
- $\boldsymbol{S}_1 \subseteq \boldsymbol{S}_2 \quad$ iff $\quad \forall \boldsymbol{x} \in \boldsymbol{S}_1.\ \boldsymbol{x} \in \boldsymbol{S}_2$
- $\boldsymbol{S}_1 \subset \boldsymbol{S}_2 \quad$ iff $\quad \boldsymbol{S}_1 \subseteq \boldsymbol{S}_2$ and $\boldsymbol{S}_1 \neq \boldsymbol{S}_2$, e.g., $\{1, 2\} \subset \{1, 2, 3\}$, $\quad \{1, 2\} \not\subset \{1, 2\}$
- $|\boldsymbol{S}|$: the number of elements in set $\boldsymbol{S}$ (for finite $\boldsymbol{S}$)
- $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are *disjoint* iff $\boldsymbol{S}_1 \cap \boldsymbol{S}_2 = \emptyset$

Construction of Sets

Union, intersection, and difference:

$$S_1 \cup S_2 = \{x \mid x \in S_1 \vee x \in S_2\}$$
$$S_1 \cap S_2 = \{x \mid x \in S_1 \wedge x \in S_2\}$$
$$S_1 - S_2 = \{x \mid x \in S_1 \wedge x \notin S_2\}$$

Complement (with respect to a universe U): $\overline{S} = \{x \mid x \in U \wedge x \notin S\}$

Powerset: $2^S = \mathcal{P}(S) = \{x \mid x \subseteq S\}$

Cartesian product:

$$S_1 \times S_2 = \{(x, y) \mid x \in S_1 \wedge y \in S_2\}$$

In general,

$$S_1 \times S_2 \times \cdots \times S_n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in S_i \text{ for all } 1 \leq i \leq n\}$$

Partition

When $S_1, S_2, \ldots, S_n$ are subsets of a given set S, they form a *partition* of S iff:

1. $S_1, S_2, \ldots, S_n$ are mutually disjoint:

$$\forall i, j.\ i \neq j \Rightarrow S_i \cap S_j = \emptyset$$

2. $S_1, S_2, \ldots, S_n$ cover S:

$$\bigcup_{1 \leq i \leq n} S_i = S$$

3. none of the S_i is empty: $\forall i.\ S_i \neq \emptyset$.

Inductive Definitions

Most of the sets we study in this course are *inductively defined*.

Example (Inductive Definition of Trees)

A set of trees is defined as follows:

1. (Basis) A single node (called *root*) is a tree.
2. (Induction) If $T_1, T_2, \ldots, T_k$ are *disjoint* trees, then the following is also a tree:
   2.1 Begin with a new node N, which is the root of the tree.
   2.2 Add edges from N to the roots of each of the trees $T_1, T_2, \ldots, T_k$.

Example (Inductive Definition of Arithmetic Expressions)

A set of arithmetic expressions is defined as follows:

- (Basis) Any number or letter (i.e., a variable) is an expression.
- (Induction) If E and F are expressions, then so are $E + F$, $E * F$, and (E).

Inductive Proofs

Induction is used to prove properties about inductively defined sets. Let S be an inductively defined set, and let $P(x)$ be a property of x. To show that $\forall x \in S.\ P(x)$, it suffices to show that:

1. (Base case) Show $P(x)$ for all basis elements $x \in S$.
2. (Inductive case) For each inductive rule using elements $x_1, \ldots, x_k$ of S to construct an element x, show that

   $$\text{if } P(x_1), \ldots, P(x_k) \text{ then } P(x).$$

   Here $P(x_1), \ldots, P(x_k)$ are the *induction hypotheses*.

Inductive Proofs: Example (Trees)

Theorem

Every tree has one more node than it has edges.

Proof. Formally, we prove $P(T) =$ "if T is a tree with n nodes and e edges, then $n = e + 1$".

- **Base case:** T is a single node. Then $n = 1$ and $e = 0$, so $n = e + 1$ holds.
- **Inductive case:** T is built from a root node N and k smaller trees $T_1, T_2, \ldots, T_k$.
    - *Induction hypothesis:* $P(T_i)$ holds for $i = 1, \ldots, k$; i.e., if T_i has n_i nodes and e_i edges, then $n_i = e_i + 1$.
    - *To show:* if T has n nodes and e edges, then $n = e + 1$.

Inductive Proofs: Example (Trees, cont.)

The nodes of T are node N and all nodes of the T_i's, so $n = 1 + n_1 + \cdots + n_k$. The edges of T are the k edges added in the inductive step, plus the edges of the T_i's, so $e = k + e_1 + \cdots + e_k$. Then:

$$
\begin{aligned}
n &= 1 + n_1 + \cdots + n_k && \text{(def. of } n\text{)} \\
&= 1 + (e_1 + 1) + \cdots + (e_k + 1) && \text{(induction hypothesis)} \\
&= 1 + k + e_1 + \cdots + e_k \\
&= 1 + e && \text{(def. of } e\text{)}
\end{aligned}
$$

Hence $n = e + 1$, completing the induction. $\square$

Inductive Proofs: Example (Expressions)

Theorem

Every arithmetic expression has an equal number of left and right parentheses.

Proof. Formally, $P(G) =$ "if G has l left parentheses and r right parentheses, then $l = r$".

- **Base case:** G is a number or a variable, in which case $l = r = 0$.
- **Inductive case:** G is constructed from smaller expressions. There are three cases.

Inductive Proofs: Example (Expressions, cont.)

Write l_E and r_E for the numbers of left and right parentheses in E (and similarly for F and G).

Case $G = E + F$:

- *Induction hypothesis:* the statement holds for the smaller expressions: $l_E = r_E$ and $l_F = r_F$.
- *To show:* $l_G = r_G$:

$$\begin{aligned} l_G &= l_E + l_F \\ &= r_E + r_F \quad \text{(I.H.)} \\ &= r_G \end{aligned}$$

Case $G = E * F$: similar.

Case $G = (E)$: the added parentheses contribute one left and one right parenthesis, so $l_G = l_E + 1 = r_E + 1 = r_G$. □

Summary

- Sets: definition, notations, constructions, partitions.
- Inductive definitions and inductive proofs.